COMBINATORICA
Bolyai Society – Springer-Verlag

COMBINATORICA **21** (3) (2001) 417–443

TOTALLY ODD K_4-SUBDIVISIONS IN 4-CHROMATIC GRAPHS

CARSTEN THOMASSEN

Received May 26, 1998

We prove the conjecture made by Bjarne Toft in 1975 that every 4-chromatic graph contains a subdivision of K_4 in which each edge of K_4 corresponds to a path of odd length. As an auxiliary result we characterize completely the subspace of the cycle space generated by all cycles through two fixed edges. Toft's conjecture was proved independently in 1995 by Wenan Zang.

1. Introduction

Hajós's conjecture says that every k-chromatic graph contains a subdivision of the complete graph K_k. It was verified for $k=4$ by Hadwiger [4], Dirac [3] and that result also follows from an earlier result of Wagner [13]. Hajós's conjecture was disproved for each $k \geq 7$ by Catlin [2]. For $k=4$, Toft [12] suggested the extension mentioned in the abstract.

Partial results on Toft's conjecture are surveyed in [6]. A recent elegant argument based on Kempe chains due to Jensen and Shepherd [5] verifies the conjecture for 3-degenerate graphs, that is, graphs in which each subgraph has a vertex of degree at most 3. A general result in [10] says that, for every natural number k, there exists a natural number $f(k)$, such that every graph of chromatic number at least $f(k)$ contains a subdivision of K_k such that each edge in K_k corresponds to a path in the subdivision of any prescribed parity. As pointed out in [10] Toft's conjecture is particularly interesting in this connection.

Mathematics Subject Classification (2000): 05C15

The strategy of the present proof is the following. First we reduce the problem to the 3-connected case and we focus on a vertex v_0 in any 3-connected, 4-chromatic graph G. We let $N(v_0)$ denote the subgraph of G induced by the neighbors of v_0. If $N(v_0)$ has two independent edges e_1, e_2, then we try to find an odd cycle C through e_1 and e_2. Then C together with v_0 and three edges from v_0 to C form a totally odd K_4-subdivision, that is a K_4-subdivision in which each edge of K_4 corresponds to an odd path in the subdivision. In the following we call such K_4-subdivision a TOKS. Moreover, if the three paths in the TOKS incident with v_0 have length 1, then we say that v_0 is of *type* 1 in G. In order to establish the existence of C (in a minimal counterexample to Toft's conjecture) we therefore first consider the general problem of finding an odd cycle through two prescribed independent edges e_1, e_2 in a graph H. For this we consider the cycle space $Z(H)$. Bondy and Lovász [1] showed that, if A is a set of k vertices in H and H is $(k+1)$-connected, then the cycle space $Z(H)$ is generated by all those cycles in H which contain A. So, if H is nonbipartite, then H has an odd cycle through A. McCuaig and Rosenfeld [8] proved that, if H is 3-connected and e_1, e_2 are two edges in H such that $H - \{e_1, e_2\}$ is nonbipartite, then H has a cycle through $\{e_1, e_2\}$ of any prescribed parity. They also pointed out that their result cannot be derived in the same way as the parity result of Bondy and Lovász [1] since the cycle space $Z(H, e_1, e_2)$ generated by the cycles in H through e_1, e_2 is distinct from $Z(H)$. However, we calculate the dimension of $Z(H, e_1, e_2)$ for any graph H and we show that it is big enough to imply not only the result of McCuaig and Rosenfeld but also its extension to subdivisions of 3-connected graphs. We also get a polynomially bounded algorithm for finding an odd cycle through any two prescribed edges. A third immediate consequence is Lovász's result [7] that a 3-connected graph has a cycle through any set of three independent edges that does not separate the graph.

Returning to the above smallest counterexample G to Toft's conjecture, we must also treat the case where $N(v_0)$ has no two independent edges. We may assume that $N(v_0)$ has no 3-cycle. Then $N(v_0)$ has a vertex w_0 such that $N(v_0) - w_0$ has no edge. We delete v_0 and identify all vertices of $N(v_0) - w_0$ into a new vertex v_1. The resulting graph G_1 is 4-chromatic. By the minimality of G, the graph G_1 contains a TOKS. We seek a TOKS in G_1 such that v_1 is *of type* i, that is v_1 has degree i where i is 0 or 2. Clearly, this would imply that v_0 is of type 0 or 2 in G. If the TOKS in G_1 cannot be chosen such that v_1 is of type 0 or 2, then we would like v_1 to be of type 1. This is not always possible. However, we demonstrate the existence of a closely related configuration which guarantees the existence of a TOKS.

2. Terminology and preliminaries

If G is a graph, then $V(G)$ and $E(G)$ denote the sets of vertices and edges, respectively. An edge $e=xy$ is said to *join* x and y, and x, y are the *ends* of e. We also say that x and y are *neighbors*. If A, $B \subseteq V(G)$, and $x \in A$, $y \in B$ we also say that e joins A and B. A *path* $P: x_0x_1 \dots x_n$ is a graph with distinct vertices x_0, x_1, $\dots$, x_n and edges x_0x_1, x_1x_2, $\dots$, $x_{n-1}x_n$. We say that x_0 and x_n are the *ends* of P and that P is a path from x_0 to x_n. If $A \subseteq V(G)$, $B \subseteq V(G)$, $A \cap B = \emptyset$, then *a path from A to B* is a path P from a vertex x_0 in A to a vertex x_n in B such that $V(P) \cap (A \cup B) = \{x_0, x_n\}$. The *parity* of P is the parity of its length n. If we add the edge x_nx_0 to P, then we obtain a *cycle* or an $(n+1)$-*cycle* C. The *parity* of C is the parity of its length $n+1$. If a, b, c, d, $\dots$ are vertices on C, then $C[a,b]$ denotes the path in C which has a, b as ends and contains none of c, d, $\dots$. (Sometimes we abuse this notation slightly when it is clear which path from a to b in C we are thinking of.) Two edges e_1, e_2 are *independent* if they have no end in common. The number of neighbors of a vertex x in a graph G is the *degree* of x. We denote by $N(x)$ the subgraph induced by all neighbors of x. If $A \subseteq V(G) \cup E(G)$, then $G-A$ is the graph obtained by deleting A and all edges incident with vertices of A. If A consists of one element f we write $G-f$ instead of $G-\{f\}$. If H is a subgraph of G and v is a vertex in G but not in H, then $H+v$ denotes H, v and all edges in G from v to H. We also write $G-H$ instead of $G-V(H)$.

If H is a graph, then a *subdivision* of H is a graph obtained from H by inserting vertices of degree 2 on the edges of H. $V(H)$ is the set of *branch vertices* of the subdivision. An *odd subdivision* of H is a subdivision in which each edge of H becomes an odd path. A TOKS is an odd subdivision of K_4. We shall extend the notion of an odd subdivision slightly. If v is a vertex of H and we replace v by two vertices v_1, v_2 such that each edge vu becomes an edge v_1u or v_2u and, in addition, we add an even v_1-v_2 path, then the new graph is also called an odd subdivision of H. Note that the property of containing (respectively not containing) a TOKS is preserved under the operation of forming an odd subdivision.

A vertex v_0 is of *type* i ($i=0$ or 2) in G if G has a TOKS in which v_0 has degree i. Also, v_0 is of *type* 1 if G has a TOKS with v_0 as a branch vertex such that the three paths from v_0 in the TOKS have length 1. Finally, v_0 is of *type 3* if G has a TOKS H_1 and a subdivision H_2 of K_4 such that v_0 has degree 3 in each of H_1, H_2, and v_0 is the end of the same three edges in H_1, H_2, and the paths in H_2 starting at v_0 are even while the other three paths in H_2 are odd. The motivation for this definition is the following: If $N(v_0)$ has no edge and G_1 is obtained from $G-v_0$ by identifying $N(v_0)$ into a vertex v_1, then, if v_1 is of type 3 in the new graph, it follows that v_0 is of type 0, 2 or 3 in G. If G is a connected graph and $A \subseteq V(G) \cup E(G)$, then A *separates* G if $G - A$ is disconnected. A separating vertex is a *cutvertex*, and a separating edge is a *bridge*. G is *k-connected* if G has at least $k+1$ vertices and no separating vertex set with fewer than k vertices.

An edge e in a 3-connected graph G is *contractible* if its contraction results in a 3-connected graph G/e (possibly with double edges incident with the new vertex). We shall use the following observation [11, Proposition 3.2].

Lemma 2.1. *If e is a noncontractible edge in a 3-connected graph G with at least five vertices, then $G - e$ is a subdivision of a 3-connected graph.*

The proof of [9, Lemma 3.1] implies the following.

Lemma 2.2. *If A is a separating set of three vertices in a 3-connected graph G and H is a component of $G - A$, then G has a contractible edge which is in H or joins A to H.*

The *blocks* in a graph G are the maximal 2-connected subgraphs. An *endblock* is a block with a most one cutvertex of G. If G is a block, then the 3-*blocks* of G are obtained as follows. If possible, we write $G = G_1 \cup G_2$ where $G_1 \cap G_2 = \{x, y\}$ and $V(G_i) \setminus V(G_{3-i}) \neq \emptyset$ for $i = 1$, 2. We now think of G_1 and G_2 as disjoint graphs and add the edge xy to each of them. We continue this operation on each of G_1, G_2 until the resulting graphs, called the 3-blocks of G, are either 3-cycles or 3-connected graphs.

A *k-coloring* of a graph is a coloring of the vertices in colors 1, 2, ..., k such that no two neighbors have the same color. A graph G is *k-colorable* if it has a k-coloring. G is *k-chromatic* if it is k-colorable but not (k–1)-colorable. In that case we write $\chi(G) = k$.

Lemma 2.3. *Let G be a graph that can be written as $G = G_1 \cup G_2$ where $G_1 \cap G_2$ consists of two vertices x, y and also the edge xy if xy is present in G. Assume that each of G_1, G_2 is 3-colorable but G is not. Then $xy \notin E(G)$, and the notation can be chosen such that $\chi(G_1 + xy) = 4$ and the graph obtained from G_2 by identifying x and y has chromatic number 4. Moreover, for each $i = 1$, 2, G_i has an odd path from x to y and an even path from x to y.*

Proof. The first part follows easily because the 3-colorings of G_1 and G_2 cannot be chosen such that they agree on x and y. It is sufficient to verify the last part in the case where G is a minimal 4-chromatic graph. Then G is 2-connected. Since the colorings of $\{x, y\}$ in all 3-colorings of G_i are the same, $G_i - x$ has an odd cycle for $i = 1, 2$. Using this it is easy to find $x - y$ paths in G_i of different parity. ∎

If E is a set, then we may regard the set of subsets of E as a vector space over the field with two elements. The sum of two sets A and B is the symmetric difference $(A \cup B) \setminus (A \cap B)$. It is called the *modulo* 2 *sum* and is denoted $A+B$. If E is the edge set of a graph G, then the subspace generated by the cycles in G is called the *cycle space* of G and is denoted $Z(G)$. If G is connected and T is a spanning tree in G, then the collection of those cycles that have only one edge outside T is a basis of $Z(G)$. Hence

$$\dim Z(G) = | E(G) | - | V(G) | + 1.$$

3. Cycles through two prescribed edges.

Theorem 3.1. *Let e_1, e_2 be independent edges in a 3-connected graph G with n vertices. Let G' be obtained from G by replacing some nonbridges of $G - \{e_1, e_2\}$ by multiple edges. Let $m = |E(G')|$. Let Z denote the subspace of $Z(G')$ generated by the cycles containing e_1 and e_2. Then*

$$\dim Z = m - n.$$

Proof. (by induction on m). The theorem is easily verified for $G' = K_4$ so assume that $m > 6$ and $n > 4$. Since any cycle which contains e_1 but not e_2 is not in $Z(G')$, we have

$$\dim Z < \dim Z(G') = m - n + 1.$$

So we shall prove that

$$\dim Z \geq m - n.$$

If $G - \{e_1, e_2\}$ has a bridge e_3, then it is easy to see that

$$Z = Z(G' - e_3)$$

which has dimension $m - n$. So assume that

(3.1.1) $G - \{e_1, e_2\}$ has no bridge.

If G' has two edges e_3, e_4 which are not in G, then we let Z_1(respectively Z_2) denote the subspace of $Z(G' - e_3)$ (respectively $Z(G' - \{e_3, e_4\})$) generated by the cycles through e_1 and e_2. By the induction hypotesis,

$$\dim Z_i = m - n - i$$

for $i = 1$, 2. Since $\dim Z_1 > \dim Z_2$, G has a cycle through e_1, e_2, e_4. Similarly, G has a cycle through e_1, e_2, e_3. But then

$$\dim Z > \dim Z_1 = m - n - 1.$$

We may therefore assume that

(3.1.2) either $G' = G$ or $G' = G \cup \{e_3'\}$ where e_3' forms a 2-cycle with an edge e_3 in G.

Consider the case where G has an edge e having an end which is not an end of e_1 or e_2 such that G/e is 3-connected. Put $\delta = 1$ if $e = e_3$, and $\delta = 0$ otherwise. If G/e has no double edge whose deletion from $(G/e) - \{e_1, e_2\}$ results in a disconnected graph, then, by the induction hypothesis G'/e has $m - n - \delta$ linearly independent cycles $C_1, C_2, \ldots, C_{m-n-\delta}$ which contain e_1 and e_2. If C_i is a cycle in G', then we put $C_i' = C_i$. Otherwise, $C_i' = C_i \cup \{e\}$ is a cycle in G', $i = 1, 2, \ldots, m-n-\delta$. Now $C_1', C_2', \ldots, C_{m-n-\delta}'$ are linearly independent in Z. If $\delta = 0$ the proof is complete. On the other hand, if $\delta = 1$, that is $e = e_3$, then at least one of $C_1', \ldots, C_{m-n-1}'$ must contain e_3 because otherwise $C_1', C_2', \ldots, C_{m-n-1}'$ are linearly independent cycles and hence a basis of $Z(G' - \{e_3, e_3'\})$. Since e_3 is not a bridge in $G - \{e_1, e_2\}$ it follows that $G - \{e_2, e_3\}$ has a cycle through e_1. That cycle is not in Z and is therefore not a linear combination of $C_1', C_2', \ldots, C_{m-n-1}'$, a contradiction which shows that one of $C_1', C_2', \ldots, C_{m-n-1}'$, say C_1', contains e_3. Let C_1'' be obtained from C_1' by replacing e_3 by e_3'. Then $C_1', C_1'', C_2', \ldots, C_{m-n-1}'$ are linearly independent in Z. So we may assume that G/e has a double edge $\{e_4, e_5\}$ such that $G - \{e_1, e_2, e_4, e_5\}$ is disconnected. Then $\{e, e_4, e_5\}$ form a 3-cycle in G. Moreover, $G - \{e_1, e_2, e_4, e_5\}$ has precisely two components H_1, H_2, say, where e is in H_1. Let $e = xy$ where y is not an end of e_1 or e_2. Let $e_4 = zx$ and $e_5 = zy$ where $z \in V(H_2)$. Since G is 3-connected, and hence $G - z$ is 2-connected, none of e_1, e_2 has z as an end. Now we complete the proof by repeating the above reasoning with e_4 instead of e. (Clearly, G/e_4 is 3-connected and has a unique 2-cycle $\{e, e_5\}$ but $G - \{e_1, e_2, e, e_5\}$ is not disconnected.)

In what follows we may assume that

(3.1.3) if e is any edge with at most one end which is also an end of e_1 or e_2 then G/e is not 3-connected.

Let v be an end of e_1. If G has at least two edges from v to vertices which are not ends of e_1, e_2, then let e be one such edge. Then we complete the proof by applying induction to $G' - e$ since $G - e$ is a subdivision of a 3-connected graph by (3.1.3) and Lemma 2.1. (In that 3-connected graph, e_1 and e_2 correspond to edges which are independent since v has at least two neighbors which are not ends of e_1, e_2. Also note that there may be a bridge e_3 in $G - \{e_1, e_2, e\}$. But then it is easy to find a cycle through e_1, e_2, e_3 using that $G - v$ is 2-connected.) So we may assume that each end of e_1 (and similarly e_2) has at most one neighbor which is not an end of e_1 or e_2.

As each vertex has degree at least 3, G has a 4-cycle C containing e_1 and e_2. Moreover, there are at most 4 edges from C to $V(G)-C$.

Consider now an edge $e = xy$ with at most one end in C. By (3.1.3) G/e is not 3-connected. Hence G has a vertex z such that $G-\{x,y,z\}$ is disconnected. Combining Lemma 2.2 with (3.1.3) we conclude that C must intersect all components of $G-\{x,y,z\}$, in particular C contains z and one of x, y. Hence $G-V(C)$ has no edge. It is now easy to obtain a contradiction to (3.1.3) unless G is a wheel in which case the proof is easily completed. ∎

Corollary 3.2. (Lovász [7]). *If e_1, e_2, e_3 are independent edges in a 3-connected graph G such that $G-\{e_1,e_2,e_3\}$ is connected, then G has a cycle through e_1, e_2, e_3.*

Proof. Let $n = |V(G)|$ and $m = |E(G)|$. Let G' be obtained from G by adding an edge with the same ends as e_3. Then the cycle space generated by those cycles of G' which contain e_1 and e_2 has dimension $m+1-n$, by Theorem 3.1. As the cycle space of $G-e_3$ has dimension $m-n$, Corollary 3.2 follows. ∎

Corollary 3.3. *Let e_1, e_2 be two independent edges in a 3-connected graph G. Then every cycle in $G-\{e_1,e_2\}$ is a modulo 2 sum of cycles containing e_1 and e_2.*

Proof. Let Z (respectively Z') be the subspace of $Z(G)$ generated by those cycles that contain both (respectively both or none) of e_1, e_2. Then

$$Z \subseteq Z' \neq Z(G)$$

As $\dim Z = \dim Z(G) - 1$, it follows that $Z = Z'$. ∎

Corollary 3.4. *Let e_1, e_2 be two independent edges in a 3-connected graph G. Let G' be any subdivision of G. (If e_i becomes a path, then we let e_i denote any edge of that path, $i=1$, 2.) If $G'-\{e_1,e_2\}$ has an odd cycle, then G' has an odd cycle containing e_1, e_2, and G' has an even cycle containing e_1, e_2.*

Proof. Let C be an odd cycle in $G'-\{e_1,e_2\}$. By Corollary 3.3 C is the modulo 2 sum of cycles $C_1, C_2, \ldots, C_q$ in G' containing e_1 and e_2. Since C is odd, at least one of $C_1, C_2, \ldots, C_q$ must be odd. Since C contains none of e_1, e_2, q is even. Hence at least one of $C_1, C_2, \ldots, C_q$ must be even. ∎

The special case of $G' = G$ in Corollary 3.4 was obtained by McCuaig and Rosenfeld [8].

Consider the following three questions, given two edges e_1, e_2 in a graph G.

Question 1. Does G contain an odd cycle through e_1 and e_2?

Question 2. Does G contain an even cycle through e_1 and e_2?

Question 3. Do all cycles through e_1 and e_2 have the same parity?

If the answer to Question 3 is affirmative, we say that (e_1, e_2) is a *parity pair* in G.

Corollary 3.5. *There are polynomially bounded algorithms for each of Questions 1, 2 and 3.*

Proof. It is easy to see that Questions 1 and 2 have the same complexity. Since it is easy to find some cycle through e_1 and e_2 and decide its parity, Question 3 also has that complexity.

We now focus on Question 3. It suffices to consider the case where G is a 2-connected graph. If G is a subdivision of a 3-connected graph H, then we consider $G' = G - \{e_1, e_2\}$. If G' is bipartite, then clearly (e_1, e_2) is a parity pair in G. If $G' - \{e_1, e_2\}$ is nonbipartite and e_1, e_2 correspond to independent edges in H, then Corollary 3.4 implies that (e_1, e_2) is not a parity pair. If the edges f_1 and f_2 in H corresponding to e_1 and e_2, respectively, are incident with the same vertex v, then (e_1, e_2) is a parity pair if and only if all paths in $G - v$ connecting the other two ends v_1, v_2 of f_1 and f_2, respectively, have the same parity. This can be tested as follows: Let P be a path in $G - v$ (not $H - v$) between v_1 and v_2, and let $B_1, B_2, \ldots, B_k$ be the blocks of $G - v$ that intersect $E(P)$. Now e_1, e_2 is a parity pair if and only if each of $B_1, B_2, \ldots, B_k$ is bipartite.

We are left with the case that G is not a subdivision of a 3-connected graph. Then $G = G_1 \cup G_2$ when $V(G_1) \cap V(G_2) = \{x, y\}$ and $E(G_1) \cap E(G_2)$ is either empty or the edge xy, and $V(G_{3-i}) \setminus V(G_i) \neq \emptyset$ for $i = 1$, 2, and none of G_1, G_2 is just a path from x to y. We now consider $G_i \cup \{xy\}$ for $i = 1$, 2. If $e_i \in G_i$ for $i = 1$, 2, then Question 3 reduces to the question: Are both the pairs (e_1, xy) and (xy, e_2) parity pairs in $G_1 \cup \{xy\}$ and $G_2 \cup \{xy\}$, respectively? If $e_i \in E(G_1)$ for $i = 1$, 2, then we consider $G_1 \cup \{xy\}$ instead of G. If all paths in G_2 from x to y are even, we subdivide xy. If some path in G_2 from x to y is odd and some is even, we call xy a *flexible* edge. If G_2 has a flexible edge, then xy will automatically be called flexible. Continuing like this we reduce Question 3 to the case where G is a subdivision of a 3-connected graph. If there are flexible edges in the graph, then we also need to check if there is a cycle containing e_1, e_2, and some flexible edge (in which case (e_1, e_2) is not a parity pair). That can be checked using Corollary 3.2. ∎

4. Configurations containing a TOKS.

In this section we describe some configurations in which we can always find a TOKS. We shall use these results in the main proof. All statements in this section concern finite problems and some readers might find it convenient to check them by computer. However, we include complete proofs.

Let C be a cycle and let x be a vertex in C. We add a vertex y joined to x only. We also add a new vertex v_0. We shall now define an *i-extension* from y to C, v_0 for $i=1$, 2, 3.

A 1-*extension* is simply an edge from y to $C-x$. A 2-*extension* is a new vertex u together with an edge from u to y and two edges from u to C. (Possibly u is joined to x.) Finally a 3-*extension* from y to C, v_0 consists of two new vertices u_1, u_2 and an edge from u_i to y, an edge from u_i to v_0, and an edge from u_i to C for $i=1$, 2, such that u_1 and u_2 have different neighbors on C. If x_1, x_2, x_3 are prescribed vertices on C, where $x=x_1$, then we define a *weak* 2-*extension* from y to C, v_0 as a new vertex u joined to y and to a vertex u' in $(C[x_1,x_2]\cup C[x_1,x_3])-\{x_2,x_3\}$ such that $C[x_1,u']$ is odd.

In the proof indicated in the Introduction we identify at one stage $N(v_0)$ (or $N(v_0)$ except one vertex) into a new vertex v'. If $N(v')$ contains two independent edges we try to prove that v' is type 1 in the new graph. This results (in G) in an odd cycle C with three vertices x_1, x_2, x_3 and three vertices y_1, y_2, y_3 in $N(v_0)$ joined to x_1, x_2 and x_3 respectively. Then we try to find an odd subdivision of an i-extension ($i=1$, 2 or 3) from each of y_1, y_2, y_3. Finally, we use the results of this section to find a TOKS in the resulting configuration.

Proposition 4.1. *Let x_1, x_2, x_3 be vertices on an odd cycle C such that $C[x_1,x_2]$, $C[x_2,x_3]$ and $C[x_3,x_1]$ are odd. We add vertices y_1, y_2, y_3 such that y_i is joined to x_i for $i=1$, 2, 3. We also add a vertex v_0 joined to y_1, y_2 and y_3. Finally, we add a path P between y_2 and y_3 and in addition a* 1-extension *or a* 2-extension *or a* 3-extension *or a* weak 2-extension *(disjoint from P) from y_1 to C, v_0. Then v_0 is of type* 0, 2 *or* 3 *in the resulting graph G.*

Proof. If P is even, then C can be extended to a TOKS having y_2, x_1, x_2, x_3 as branch vertices in which case v_0 is of type 2. So assume that P is the edge y_2y_3. If G has an edge y_1w from y_1 to C, then G has a TOKS with branch vertices y_1, x_1, w and one of x_2, x_3 unless w is in $C[x_1,x_2]\cup C[x_1,x_3]$ and $C[x_1,w]$ is even. But in that case the unique cycle of $G-v_0$ containing y_1, y_2, y_3 shows that v_0 is of type 1 and hence also of type 3. We argue similarly if we have added a weak 2-extension from y_1 to C, v_0.

So we may assume that either G has a vertex u_1 joined to y_1 and to two distinct vertices w_1, w_2 of C, or else G has two vertices u_1, u_2 both joined to y_1 and v_0, and also there are edges u_iw_i such that $w_i \in V(C)$ ($i=1$, 2), and $w_1 \neq w_2$.

If $w_1 \in C[x_2,x_3]-\{x_2,x_3\}$, then G has a TOKS with branch vertices v_0, y_1, y_2, y_3 showing that v_0 is of type 3. So assume that both w_1 and w_2 are in $C[x_1,x_2] \cup C[x_1,x_3]$. If $C[x_1,w_1]$ is odd, then we replace that path by $x_1y_1u_1w_1$ and we also replace $C[x_2,x_3]$ by $x_2y_2y_3x_3$. So assume that both $C[x_1,w_1]$ and $C[x_1,w_2]$ are even.

If u_2 is not present, then we replace $C[w_1,w_2]$ by the path $w_1u_1w_2$ and extend the resulting cycle to a TOKS with branch vertices y_1, u_1, one of x_1, w_1, w_2, and one of x_2, x_3. On the other hand, if u_2 is present, then we replace $C[x_2,x_3]$ by $x_2y_2y_3x_3$, and $C[w_1,w_2]$ by $w_1u_1y_1u_2w_2$, and we extend the resulting cycle to a TOKS with branch vertices v_0, u_1, y_2, y_3 showing that v_0 is of type 1. It is easy to see that v_0 is also of type 3. ∎

Proposition 4.2. *Let x_1, x_2, x_3 be vertices on a cycle C such that $C[x_1,x_2]$, $C[x_2,x_3]$, $C[x_3,x_1]$ are odd. For each $i=1$, 2, 3 we add a vertex y_i which we join to x_i. We also add a vertex v_0 joined to y_1, y_2, y_3. For each $i=1$, 2, 3 we add either a new edge from y_i to C (a 1-extension), or else we add a new vertex u_i and join u_i to y_i and to two vertices of C (a 2-extension), or else we add two new vertices $u_{i,1}$, $u_{i,2}$ and the edges $u_{i,1}v_0$, $u_{i,2}v_0$, $u_{i,1}y_i$, $u_{i,2}y_i$, and we also add an edge from $u_{i,j}$ to C ($j=1$, 2) such that $u_{i,1}$ and $u_{i,2}$ have different neighbors on C (a 3-extension). Then v_0 is of type 0, 2, or 3 in the resulting graph G. The same conclusion holds if one (but only one) of the above extensions is replaced by a weak 2-extension.*

Proof. We consider first the case where v_0 has degree 3 in G. Note that $V(C) \cup \{y_1,y_2,y_3,v_0\}$induces a K_4-subdivision in which all paths from v_0 are even. So it suffices to prove that G has a TOKS. Suppose therefore (reductio ad absurdum) that G has no TOKS. Then

(4.1.1) if $\{i,j,k\}=\{1,2,3\}$,
then y_i is not joined to any vertex of $C[x_j,x_k]$.

For if y_i is joined to w in $C[x_j,x_k]$, then $C \cup (x_iy_iw)$ union one of the paths $y_iv_0y_jx_j$ or $y_iv_0y_kx_k$ is a TOKS.

(4.1.2). If y_i is joined to w on C, then $C[x_i,w]$ is even, $i=1$, 2, 3.

For if $C[x_i,w]$ is odd, then the cycle $C[x_i,w] \cup (x_iy_iw)$ can be extended to a TOKS with branch vertices y_i, x_i, w and a vertex in $\{x_1,x_2,x_3\} \setminus \{x_i\}$.

(4.1.3) There exists an $i \in \{1,2,3\}$ such that y_i is not joined to $C-x_i$, and such that there is no path $y_i u_i w$ such that w is in $C[x_j,x_i] \cup C[x_i,x_k] - \{x_j,x_k\}$ and $C[w,x_i]$ is odd (where $\{i,j,k\}=\{1,2,3\}$).

For if (4.1.3) were false, then $G-v_0$ has a cycle through y_1, y_2, y_3 which can be extended to a TOKS with branch vertices v_0, y_1, y_2, y_3.

By (4.1.3) we may assume that u_1 exists. Let w_1 be a neighbor of u_1 on $C-x_1$. If u_1 has another neighbor on C, we denote that by w_2. If w_i is in $C[x_1,x_2] \cup C[x_1,x_3] - \{x_2,x_3\}$, then $C[x_1,w_i]$ is even for $i=1$, 2 by (4.1.3).

(4.1.4). If w_2 exists and both of w_1, w_2 are on $C[x_2,x_3]$ (in the order x_2, w_1, w_2, x_3 say), then $C[x_2,w_1]$ and $C[w_2,x_3]$ are even.

For if $C[x_2,w_1]$ and $C[w_2,x_3]$ are both odd, then $w_2u_1w_1\ldots w_2$ can be extended to a TOKS with branch vertices x_2, w_1, w_2, u_1. And if $C[x_2,w_1]$ is even and $C[w_2,x_3]$ is odd, then we replace in C the even path $C[w_1,w_2]$ by the path $w_1u_1w_2$ and extend the resulting cycle to a TOKS with branch vertices y_1, x_1, x_2, u_1. This proves (4.1.4).

(4.1.5). If w_2 exists, then we may assume that w_2 is in $C[x_2,x_3]$.

For if w_1, w_2 are both in $C[x_2,x_1] \cup C[x_1,x_3] - \{x_2,x_3\}$, then both of $C[w_1,x_1]$, $C[w_2,x_1]$ are even by (4.1.3). We replace in C the even path $C[w_1,w_2]$ by the path $w_1u_1w_2$ and extend the resulting cycle to a TOKS with branch vertices y_1, u_1, one of x_1, w_1, w_2, and one of x_2, x_3. This proves (4.1.5).

(4.1.6). If w_2 exists, then both of w_1, w_2 are in $C[x_2,x_3]$.

For otherwise, w_2 is in $C[x_2,x_3]$ (by (4.1.5)) and w_1 is in $C[x_2,x_1] \cup C[x_1,x_3]$, and $C[w_1,x_1]$ is even. We may also assume that $C[w_2,x_3]$ is even. Then the cycle $w_2\ldots x_3\ldots w_1u_1w_2$ can be extended to a TOKS with branch vertices y_1, u_1, x_3, and one of x_1, w_1. This proves (4.1.6).

We may choose the notation such that also

(4.1.7) u_2 exists.

For suppose (reductio ad absurdum) that both y_2 and y_3 are joined to vertices of C. If w_1 is in $C[x_3,x_1] \cup C[x_1,x_2]$, then $C[w_1,x_1]$ is odd, by (4.1.6), and the unique cycle in $G-v_0$ containing y_1, y_2, y_3 can be extended to a TOKS with branch vertices v_0, y_1, y_2, y_3. By (4.1.4) and (4.1.6), we may assume that w_1, w_2 are in $C[x_2,x_3]$ and that $C[w_2,x_3]$ is even. Now the cycle $C[w_1,w_2] \cup (w_2u_1w_1)$ can be extended to a TOKS with branch vertices y_3, u_1, and two vertices in $C[w_1,w_2]$ unless y_3 is joined to a vertex in $C[x_2,w_1]$. But then the cycle $x_1y_1u_1w_1\ldots w_2\ldots x_3\ldots x_1$ can be extended to a TOKS with branch vertices x_2, x_1, y_1, x_3. This contradiction proves that u_2 exists.

We also prove that

(4.1.8) u_3 exists.

For suppose (reductio ad absurdum) that u_3 does not exist. If w_2 exists, then w_1 and w_2 are both in $C[x_2,x_3]$ and we obtain a contradiction as in the proof of (4.1.7). Since both u_1 and u_2 exist and since at most one of the extensions is a weak 2-extension, we may assume that in fact w_2 exists. This proves (4.1.8).

Since there is at most one weak 2-extension, we may assume that there is no weak 2-extension from neither y_1 nor y_2. Then w_2 exists, and u_2 has two neighbors in $C[x_1,x_3]$ satisfying (4.1.4). But then the cycle $w_1 \dots w_2 u_1 w_1$ can be extended to a TOKS with branch vertices y_2, w_2, w_1, u_1. This contradiction completes the proof when v_0 has degree 3 in G. Suppose therefore that $u_{1,1}$ and $u_{1,2}$ exist.

Let w_i be the neighbor of $u_{1,i}$ in C, $i=1$, 2. If there exists a $j \in \{2,3\}$ such that x_1, w_1, x_j divide C into three odd paths, then we apply Proposition 4.1 with x_1, w_1, x_j instead of x_1, x_2, x_3. So we may assume that no such $j \in \{2,3\}$ exists. But then $w_1 \in C[x_1,x_2] \cup C[x_1,x_3]$ and $C[x_1,w_1]$ is even. Then we let $u_{1,1}$ play the role of y_1, and we think of the odd path $u_{1,1} y_1 u_{1,2} w_2$ as a single edge. We do this for each 3-extension. Now we complete the proof by applying the first part of the proof (where v_0 has degree 3). ∎

5. Toft's conjecture.

We shall prove an extension of the conjecture. For that we need the notion of a *strong TOKS* defined recursively as follows. A TOKS which consists of a cycle C together with a vertex v_0 and three edges from v_0 to C is a strong TOKS. We say that v_0 is the *distinguished vertex* of that strong TOKS. Assume now that each of H_1, H_2 is a strong TOKS with distinguished vertex v_1, v_2, respectively. Let $v_i x_i \in E(H_i)$ for $i=1$, 2. Now form a new graph H by first identifying v_1, v_2 into a vertex v_0, then delete $v_0 x_i$ ($i=1$, 2) and add the edge $x_1 x_2$. Then H is a strong TOKS with v_0 as distinguished vertex. Note that a strong TOKS contains a TOKS.

Theorem 5.1. *Let v_0 be a vertex in a graph G of chromatic number at least 4. Then either v_0 is of type 0, 2 or 3 in G or else v_0 is the distinguished vertex of a strong TOKS in G.*

Proof. Suppose (reductio ad absurdum) that Theorem 5.1 is false and let G be a counterexample to Theorem 5.1 with as few vertices as possible and (subject to that) as few edges as possible. We shall derive a number of properties of G and finally reach a contradiction. The minimality of G implies:

(5.1.1) Every proper subgraph of G is 3-colorable, and G is 3-connected.

Proof of (5.1.1). The first assertion follows from the minimality of G. Hence G is 2-connected. Suppose (reductio ad absurdum) that G has two vertices x, y such that $G-\{x,y\}$ is disconnected. Assume that $v_0=y$. (The case where $v_0 \notin \{x,y\}$ is similar and easier.) Then we write $G = G_1 \cup G_2$ where $G_1 \cap G_2$ consists of v_0, x and possibly the edge v_0x, and $V(G_{3-i}) \setminus V(G_i) \neq \emptyset$ for $i=1$, 2. The minimality of G implies that each of G_1, G_2 is 3-colorable. Then the edge v_0x is not present and the notation can be chosen such that the graph G_1' (respectively G_2') obtained from G_1 (respectively G_2) by adding the edge v_0x (respectively identifying v_0 and x) is 4-chromatic. By the minimality of G, (G_1', v_0) satisfies Theorem 5.1. So does (G, v_0) (by Lemma 2.3) unless v_0 is the distinguished vertex of a strong TOKS H_1 containing the edge v_0x. Also (G_2', v_0) satisfies Theorem 5.1. Again, we may assume that G_2' contains a strong TOKS in which v_0 is a distinguished vertex. Let H_2 denote the corresponding subgraph in G_2. If x is not in H_2, then H_2 is a strong TOKS in G. If x has degree 1 in H_2, then $H_1 \cup H_2 - v_0x$ is a strong TOKS in G. Finally, if x has degree ≥ 2 in H_2, then G has a TOKS in which x but not v_0 is a branch vertex. So v_0 is of type 0 or 2 in that case. This contradiction proves (5.1.1).

(5.1.2). If v_0 is joined to x and y, and $xy \in E(G)$ and x has degree 3 in G, then the neighbor x' of x distinct from y and v_0 is not joined to v_0.

Proof of (5.1.2). $G-v_0$ is 2-connected and nonbipartite and has therefore an odd path from y to x' which together with the path $x'xy$ and the edges from v_0 to that path forms a TOKS if v_0 is joined to x'. So x' is not joined to v_0.

(5.1.3). If v_0 is joined to x and y, where $xy \in E(G)$, $d(x)=3$, and x' is the third neighbor of x, then $x'y \notin E(G)$.

Proof of (5.1.3). Suppose (reductio ad absurdum) that $x'y \in E(G)$. Form a new graph G' by deleting x and identifying v_0 and x'. Then $\chi(G') \geq 4$ and therefore v_0 is the distinguished vertex in a strong TOKS H' in G'. Let H be the corresponding subgraph in G. If x' has degree 0 in H, then v_0 is the distinguished vertex of a strong TOKS in G. If x' has degree ≥ 2 in H, then it is easy to find a TOKS in G where x' but not v_0 is a branch vertex, i.e., v_0 is of type 0 or 2. So assume that x' has degree 1 in H. If $y \notin V(H)$ then we add the 3-cycle $xyx'x$ to H and obtain a strong TOKS with v_0 as branch vertex. So assume that $y \in V(H)$. Let z be the neighbor of x' in H, and let C be the cycle in $H-\{v_0, x'\}$ containing z. Let z_1, z_2 be the other vertices on C which have degree 3 in H. If $y \in V(C)$, then G has a TOKS with branch vertices x', z, y, and one of z_1, z_2 unless y is in $C[z, z_1]$, and

$C[z,y]$ is even. But then v_0 is the distinguished vertex of the strong TOKS obtained from H by replacing the edge $x'z$ by v_0y. If $y \notin V(C)$, then G has a TOKS with branch vertices x', z, z_1, z_2 unless y is a neighbor of one of z_1, z_2 in H. As v_0 is joined to y, it follows that G has a strong TOKS with v_0 as distinguished vertex. This proves (5.1.3).

(5.1.4). If v_0 is joined to x and y, and $xy \in E(G)$, and x has degree 3 in G, then y has degree at least 4 in G.

Proof of (5.1.4). Let x' be as in (5.1.2). If y has degree 3 in G, we define y' as the third neighbor of y. By (5.1.2), v_0 is joined to neither of x', y'. By (5.1.3), $x' \neq y'$. But then $G-\{x,y\}+x'y'$ is a smaller counterexample to Theorem 5.1, a contradiction which proves (5.1.4).

(5.1.5). If $N(v_0)$ has two independent edges e_1, e_2, then e_1, e_2 do not belong to the same 3-block of $G-v_0$.

Proof of (5.1.5). Suppose (reductio ad absurdum) that e_1, e_2 belong to the same 3-block B of $G-v_0$. Consider any edge e in B which is also in another 3-block of $G-v_0$. Then $G-v_0$ has a path P_e which has the same ends as e and which has only these ends in common with B. If every such path P_e is even, then we subdivide e in B once. If P_e can be chosen to be even and can also be chosen to be odd, then we call e *remarkable*. We may assume that B has no cycle containing both e_1, e_2 and a remarkable edge. For otherwise, $G-v_0$ has an odd cycle through e_1, e_2 and hence v_0 is of type 1 in G. In particular, a remarkable edge cannot have an end in common with e_1 or e_2 because a 3-connected graph has a cycle through any three edges which are not independent and do not form a star. So, if e is remarkable, then $B-\{e_1,e_2,e\}$ is disconnected, by Corollary 3.2. In other words, e is a bridge in $B-\{e_1,e_2\}$. Since B is 3-connected, $B-\{e_1,e_2\}$ has at most one bridge, and hence B has at most one remarkable edge.

Consider first the case where B has no remarkable edge. Let B' be obtained by subdividing those edges e for which P_e is even. Since $G-v_0$ has no odd cycle through e_1 and e_2, B' has no odd cycle through e_1 and e_2. By Corollary 3.4, $B'-\{e_1,e_2\}$ is bipartite with bipartition V_1, V_2 say. This bipartition can be extended to a bipartition of $(G-v_0)-\{e_1,e_2\}$ because B has no remarkable edge. Since G is not 3-colorable, the notation may be chosen such that e_1 joins two vertices of V_1 and v_0 is also joined to some vertex v in V_2. Since B is 3-connected, $B-e_2$ is 2-connected and so $G-v_0-e_2$ contains a cycle which contains e_1 and v. Now $C+v_0$ contains a TOKS, a contradiction. We may therefore assume that B has precisely one remarkable edge e_3. As mentioned earlier, e_1, e_2, e_3 are pairwise independent.

By Corollary 3.2, $B-\{e_1,e_2,e_3\}$ is disconnected. Since B is 3-connected, the six ends of e_1, e_2, e_3 are distinct, and the two components B_1, B_2 of $B'-\{e_1,e_2,e_3\}$ are 2-connected.

If one of B_1, B_2 is nonbipartite, then $B'-e_3$ is a 2-connected nonbipartite graph and has therefore an odd cycle through e_1, e_2. So we may assume that $B_1 \cup B_2$ is bipartite with bipartition V_1, V_2. Since $G-v_0$ has no odd cycle through e_1 and e_2, also $B'-e_3$ is bipartite so we may assume that e_i joins a vertex of V_1 with a vertex of V_2 for $i=1$, 2. Also, we may assume that $e_1=x_1y_1$ and $e_2=x_2y_2$ where x_1, $x_2 \in V(B_1)$, $x_1 \in V_1$. If v_0 has a neighbor v in $V_1 \cap V(B_2)$, then it is easy to see that $G-v_0-e_2$ has an odd cycle through e_1 and v. That cycle can be extended to a TOKS with branch vertices x_1, y_1, v, v_0. So assume that v_0 has no neighbor in $V_1 \cap V(B_2)$. In particular, $y_2 \in V_2 \cap V(B_2)$. Let y_3 denote the end of e_3 in B_2. If $y_3 \in V_2$, then we form a new graph G' by first deleting $V_1 \cap V(B_2)$, and then identifying y_1 and y_2 into a single vertex. Clearly $\chi(G') \geq 4$. But then v_0 is of type 0, 2 or 3, or v_0 is the distinguished vertex of a strong TOKS in G'. Since B is 3-connected it follows that the same holds in G, a contradiction. So we may assume that $y_3 \in V_1 \cap V(B_2)$. Now we form a new graph G'' by first identifying $V_1 \cap V(B_2)$ into y_3 and then deleting all vertices of $V_2 \cap V(B_2)$ except y_1 and y_2. Clearly, $\chi(G'') \geq 4$. Since B is 3-connected, G'' is smaller than G. So G'' is a smaller counterexample to Theorem 5.1, a contradiction which proves (5.1.5).

(5.1.6) If v_0 is joined to x and y, where $xy \in E(G)$, then $G-\{v_0,x,y\}$ is connected. If, in addition, x has degree 3 and x' is the third neighbor of x, then also $G-\{v_0,x',x,y\}$ is connected.

Proof of (5.1.6). The first assertion is trivial because every proper subgraph of G is 3-colorable whereas G is not. Suppose now (reductio ad absurdum) that the last assertion fails. Then we write $G=M_1 \cup M_2$ where $M_1 \cap M_2$ is the cycle v_0yxv_0 and the edge xx', and $V(M_i) \setminus V(M_{3-i}) \neq \emptyset$ for $i=1$, 2. By (5.1.1), each of M_1, M_2 is 3-colorable. We may choose the notation such that M_1+yx' is not 3-colorable. Hence M_1+yx' contains a TOKS satisfying the conclusion of Theorem 5.1. Since G has no such TOKS, all paths in M_2-v_0 from y to x' are even. Hence M_2-v_0 is bipartite. Consider a 3-coloring of M_1. We may assume that x, y, v_0 have colors 1, 2, 3, respectively, and that x' has color 2 because M_1+yx' is not 3-colorable. That 3-coloring can be extended to M_2-v_0. This contradiction proves (5.1.6).

(5.1.7) $N(v_0)$ has no two independent edges.

Proof of (5.1.7). Suppose (reductio ad absurdum) that e_1, e_2 are independent edges in $N(v_0)$. By (5.1.5) they do not belong to the same 3-block of $G-v_0$.

We let G' denote the graph obtained from $G–v_0$ by replacing the maximal path which contains e_i and has all intermediate vertices of degree 2 by a single edge e_i', $i=1$, 2. By (5.1.4), either $e_i'=e_i$ or e_i' corresponds to a path of length 2 in $G-v_0$. Note that e_1', e_2' cannot form a double edge because v_0 is not joined to the vertex x' in (5.1.2). By (5.1.3) neither of e_1', e_2' is contained in a cycle of length 2.

Now we form the 3-blocks of G'. Let B be a 3-block of G' containing e_1', and let e_2'' be the unique edge of B corresponding to e_2'. That is, $(G–v_0)\cup\{e_2''\}$ contains a cycle C which contains e_2 and e_2'' and has only e_2'' and the ends of e_2'' in common with B. By (5.1.6), e_1' is in only one 3-block of G'. Hence $e_2''\neq e_1'$. Since $G-v_0$ has no odd cycle through e_1 and e_2, we may assume that all cycles C above have the same parity.

Case 1. e_1' and e_2'' are incident with the same vertex y. That is $e_1'=yx$ and $e_2''=yz$. (This includes the case where B is the 3-cycle $xyzx$.) Then either $e_1=xy$ or else there is a vertex x' such that $e_1=xx'$ or $e_1=x'y$, in which case we say that x' *exists*. Let H be the subgraph consisting of the union of all paths from x to z in $G-\{v_0,y\}$. Since all cycles in $G-v_0$ through e_1 and e_2 have the same parity, all paths in H from x to z have the same parity. Hence H is 2-colorable. Let H' be the subgraph of $G-v_0$ consisting of the union of all paths from y to z that intersect $H-\{y,z\}$. In other words, H' consists of H and all subgraphs in G corresponding to edges in B incident with y but not z. We also define M as the union of all paths from y to z in $G-v_0$ having only y and z in common with B. In particular, M contains e_2. We now claim

(5.1.7a). H' is not bipartite. For suppose (reductio ad absurdum) that the 2-coloring of H can be extended to H'. Then we form a new graph M' from $M+v_0$ by adding the edge yz (if all paths in H' from y to z are odd) or a path $yz'z$ (if all paths in H' from y to z are even). If z' exists, then we also add $z'v_0$ to M'.

Any 3-coloring of M' can be extended to a 3-coloring of G. Hence M' has no 3-coloring. If M' satisfies the conclusion of Theorem 5.1, then so does G. Hence $M'=G$. But then B is not a 3-block in G', a contradiction which proves (5.1.7a).

(5.1.7b) M is not bipartite.

Proof of (5.1.7b). The proof is the same as that of (5.1.7a). Instead of reaching a contradiction as in the proof of (5.1.7a) we conclude by (5.1.2) that x' exists, M is a path $yz'z$, $e_1=x'x$ and $e_2=z'z$.

Now we delete x', z' and identify v_0 and y. The resulting graphs G'' is not 3-colorable and satisfies therefore the conclusion of Theorem 5.1. so does G

unless G'' has a strong TOKS Q in which y has degree 1. Since H is bipartite, then all odd cycles of $Q - v_0$ are in subgraphs of G corresponding to edges of B incident with y. If these odd cycles are not in subgraphs corresponding to the same edge of B, then it is easy to find a strong TOKS in G where one of the odd cycles contains e_1. So assume that all odd cycles of $Q - v_0$ are in subgraphs of G corresponding to the edge yw, say, in B. If $H' - w$ has an odd cycle through e_1, then Q can be extended to a TOKS in G. So assume that the block of $H' - w$ containing e_1 is bipartite. In other words, there exists a 2-coloring of H and all those subgraphs of G corresponding to edges of B distinct from yw. If y and w have distinct colors, we add the edge yw. Otherwise we add a path $yw'w$ and the edge v_0w'. Now we complete the proof as in (5.1.7a) by applying Theorem 5.1 to the union of the subgraphs of G corresponding to the edge yw of B. This proves (5.1.7b).

(5.1.7c) G has at most one edge joining two of v_0, y, z.

Proof of (5.1.7c). We forget about the particular structure of H' and M and focus only on the fact that v_0, y, z divide G into the graphs $H' + v_0$ and $M + v_0$. (We consider these two graphs and also y, z to be "symmetric"). If v_0, y, z induces a 3-cycle, then one of these two graphs would be a counterexample to Theorem 5.1. By (5.1.5), that counterexample is smaller than G, a contradiction. If G contains the edges v_0y and v_0z, then either $(M + v_0) \cup \{yz\}$ or $(H' + v_0) \cup \{yz\}$ would be a counterexample by (5.1.7a) and (5.1.7b). If G contains yz, and v_0y say, then we may assume that the graph H'' obtained from $H' + v_0$ by adding v_0z is 4-chromatic and that the graph M' obtained from $M + v_0$ by identifying z with v_0 is 4-chromatic. (Otherwise G would be 3-colorable). So each of H'', M' satisfies the conclusion of Theorem 5.1. Then also G does unless v_0 is the distinguished vertex of a strong TOKS in both H'' and M'. Moreover z has only degree 1 in the strong TOKS in M' (since otherwise v_0 would be of type 0 or 2 in G). But then the union of these two strong TOKS's form a strong TOKS in G. This proves (5.1.7c).

(5.1.7d). v_0 is not joined to y.

Proof of (5.1.7d). Suppose (reductio ad absurdum) that v_0 is joined to y. By (5.1.7c), z is not joined to any of y, v_0. Now we identify y and z in each of $H' + v_0$, $M + v_0$. One of the resulting graphs is 4-chromatic and satisfies the conclusion of Theorem 5.1. So does G unless v_0 is the distinguished vertex of a strong TOKS corresponding to a subgraph Q_1 of G which is not a strong TOKS in G. That strong TOKS Q_1 would have to contain z but not y. That is only possible in M since $H' - y$ has no odd cycle through z. Now we identify instead z and v_0 in each of $H' + v_0$, $M + v_0$. Again we conclude that v_0 is the distinguished vertex of a strong TOKS Q_2 in one of the resulting graphs and that z has degree 1 in Q_2. (For if z has degree at least 2 in Q_2, then we get three odd paths from z to $Q_2 - v_0 - z$ and hence a TOKS in which v_0 has degree 0 or 2 by considering an odd path from z to y disjoint from $Q_2 - v_0 - z - y$.) If Q_2 is in $H' + v_0$, then $Q_1 \cup Q_2$ is a strong TOKS in G. So we may assume that both Q_1 and Q_2 are in $M + v_0$. Let yt be an edge of B corresponding to a subgraph of G which together with H forms a nonbipartite graph. Now H' has an odd cycle containing z, t, y and e_1. If the two paths in that odd cycle from z to e_1 are odd, then we extend Q_2 to a strong TOKS except if Q_2 contains y. Then we may assume that Q_2 also contains $v_0 y$ and we obtain a TOKS in which z has degree 3 and v_0 has degree 2.

So we may assume that in the odd cycle in H' through z and e_1, both paths from z to e_1 are even. Hence all paths in H' from z to y containing e_1 are odd. Since all cycles in $G - v_0$ through e_1 and e_2 are even, if follows that all paths in M from y to z containing e_2 are odd. We have previously noted that we would obtain a contradiction if one of the two graphs obtained from $H' + v_0$ by identifying z with either y or v_0 is 4-chromatic. So both of these are 3-colorable. Since $M + v_0$ is 3-colorable, there must be some 3-coloring of v_0, y, z that cannot be extended to $H' + v_0$. That must be the 3-coloring where v_0, y, z have distinct colors. In other words, $(H' + v_0) \cup \{zv_0, zy\}$ is 4-chromatic and satisfies the conclusion of Theorem 5.1. So does G because zy may be replaced by a path from z to y through e_2, and $v_0 z$ may be replaced by one of the edges from v_0 to e_2. This proves (5.1.7d).

By (5.1.7d) x' exists, and $e_1 = x'x$. Also, x' has degree 3. Now we delete x' and identify v_0 and y. The resulting graph G' is 4-chromatic and satisfies therefore the conclusion of Theorem 5.1. So does G unless v_0 (and y) is the distinguished vertex of a strong TOKS Q (in G' but not in G) in which y has degree 1 (in G). We shall try to extend this strong TOKS Q to a strong TOKS in G by adding an odd cycle through $e_1 = xx'$. By (5.1.7a) or (5.1.7b) this is possible unless Q intersects both $H' - \{y, z\}$ and $M - \{y, z\}$. (Note that if Q does not intersect $M - \{y, z\}$, then Q is contained in a subgraph of G corresponding to edges of B incident with y but not z). It follows that part of Q is in $M + v_0$ and the remaining part of Q is in a subgraph corresponding to an edge yu in B, and Q also contains the edge uz which is a cutedge in $Q - v_0$. Now G has a TOKS in which y has degree 3 and v_0 has degree 2. This completes the proof of Case 1 in (5.1.7).

Case 2. e_1' and e_2'' are independent.

Consider an edge $e = vz$ of B such that $e \neq e_2''$ and $G - \{v_0, v, z\}$ is disconnected (if such an edge e exists in B). We claim that all paths in $G-$

v_0–$(V(B)\setminus\{v,z\})$ from v to z must have the same parity. For otherwise B has no cycle through e_1', e_2'', e. In particular, e_1', e_2'', e are pairwise independent because B is 3-connected. By Corollary 3.2, $B-\{e_1',e_2'',e\}$ is disconnected with components H_1', H_2' such that e_1', e_2'', e are independent edges from H_1' to H_2'. For $i=1$, 2, we let H_i be obtained from H_i' by replacing every edge ab by the union of all paths from a to b in $G{-}v_0$ having only a and b in common with B. Since each of H_1, H_2 is 2-connected and $G-v_0$ has no odd cycle through e_1, e_2, it follows that H_i is bipartite with bipartition $V_{1,i}$, $V_{2,i}$ for $i=1$, 2. Choose the notation such that $V_{1,1}$ contains at most one end of one of e_1', e_2'', e. Now contract $V_{1,1}$ into a single vertex u and delete all vertices of $V_{2,1}$ except the ends of e_1', e_2'', e. If v_0 has a neighbor in $V_{1,1}$ then we also add the edge uv_0 and we call the resulting graph L. Clearly, $\chi(L)\geq 4$. For if L has a 3-coloring c, then that can be extended to a 3-coloring of G by first giving $V_{1,1}$ the color $c(u)$ and then color each uncolored vertex of $V_{2,1}$ by a color distinct from $c(u)$, $c(v_0)$. Also, a TOKS in L satisfying Theorem 5.1 can be modified to a similar TOKS in G because B is 3-connected and H_1 is bipartite. So $V_{1,1}=\{u\}$. But then all vertices in $V_{2,1}$ have degree 2 in B, contradicting that B is 3-connected. This proves the claim that all paths in $G-v_0-(V(B)\setminus\{v,z\})$ from v to z have the same parity. If that parity is even, then $e\notin E(G)$, and we replace e by a path of length 2. Doing this for every edge e in B, we transform B into a graph B' which is a subdivision of the 3-connected graph B.

By Corollary 3.4, $B'-\{e_1',e_2''\}$ is bipartite with bipartition V_1, V_2 say. Let $e_1'=x_1y_1$, and $e_2''=x_2y_2$. Either e_1' is an edge of G, or G contains a path $x_1z_1y_1$ where x_1, z_1 are joined to v_0, and z_1 has degree 3 in G. We say that z_1 *exists* in that case. Let H denote the subgraph of $G-v_0$ which is the union of all paths from x_2 to y_2 having only x_2 and y_2 in common with B.

Consider first

Case 2.1. $x_2\in V_2$, $y_2\in V_1$. We may assume that $H+v_0+x_2y_2$ is 3-colorable since otherwise it is a smaller counterexample to Theorem 5.1.

We may extend any 3-coloring of $H+v_0+x_2y_2$ to G unless either z_1 exists and x_1, y_1 belong to distinct sets V_1, V_2 (say $y_1\in V_1$, $x_1\in V_2$) or else z_1 does not exist and x_1, y_1 belong to the same of V_1, V_2, say V_2. All cycles in $G{-}v_0$ through e_1 and e_2 are even. Hence all cycles in $G{-}v_0{-}e_1$ through x_1 and e_2 are odd (and such cycles exist since B is 3-connected). Since v_0 is not of type 1, the paths from x_1 to e_2 in these cycles are even. Then it is easy to see that v_0 cannot be joined to any vertex x' in V_1 since any cycle in G–v_0–e_1 through e_2 and x' could be extended to a TOKS showing that v_0 is of type 1. In particular, v_0 is not joined to y_2. Now let H' be obtained from $H{+}v_0$ by identifying v_0 and y_2 and adding the edge y_2x_2. Any 3-coloring of

H' can be extended to a 3-coloring of G (by letting all vertices of V_1 have the same color as y_2 and letting all vertices of $V_2 \setminus \{z_1, y_1\}$ have the same color as x_2 and then giving z_1 or y_1 the third color). So H' satisfies the conclusion of Theorem 5.1. Since $B - e'_1$ is 2-connected, it has two paths P_1, P_2 from y_2 to x_1 and x_2, respectively, having only y_2 in common. Using these paths and the edge $x_1 v_0$ we conclude that G contains an odd subdivision of H', and so it follows that G satisfies the conclusion of Theorem 5.1 unless v_0 is the distinguished vertex of a strong TOKS in H'. Let M denote the corresponding subgraph in $H + v_0$ (or in $H + v_0 + y_2 x_2$). If y_2 has degree 0 or ≥ 2 in M, then it is easy to see that either v_0 is the distinguished vertex of a strong TOKS in G or else that G has a TOKS having y_2 but not v_0 as a branch vertex (possibly using the paths P_1, P_2 above). So we may assume that y_2 has degree 1 in M, possibly y_2 is joined to x_2 in M. Then we add to M a cycle C in $B' - x_2$ such that C contains y_2 and e_1 and conclude that v_0 is the distinguished vertex of a strong TOKS unless y_2 is joined to x_2 in M but not in G. Then we let y'_2 be a neighbor of x_2 in B' and now we let C be a cycle in $B' - x_2$ containing y'_2 and e_1. (C may contain y_2 which will create no problem.) This cycle C exists unless all neighbors of x_2 in V_1 have degree 2 in B'. Then none of these neighbors equals y_1 (because otherwise B would be a K_3 and not a 3-connected graph and we would be back to Case 1). Now we construct H' and argue as above except that we do not include the edge $y_2 x_2$ in H'. The arguments above now work unless the new H' has a 3 -coloring where y_2 and x_2 have the same color 1 say. If z_1 does not exist, we color y_1 by 3, all vertices of $V_2 \setminus \{x_2, y_1\}$ by 2 and then successively color the vertices V_1 none of which is joined to more than two colors. If z_1 exists, then we color x_1, z_1, y_1 by 2, 3, 1, respectively, all vertices of $V_2 \setminus \{x_2\}$ by 2, and all vertices of $V_1 \setminus \{y_1, y_2\}$ by 3. This completes the discussion when x_2, y_2 belong to distinct sets V_1, V_2.

Case 2.2. Assume now that $x_2, y_2 \in V_1$. We form a new graph Q from G by deleting $V_1 \setminus \{x_2, y_2\}$ and identifying all vertices of V_2 into one vertex z_0. If v_0 is joined to some vertex of V_2, then v_0 is also joined to z_0 in Q. Since B is 3-connected, any configuration in Q satisfying the conclusion of Theorem 5.1 can be transformed to a similar configuration in G. Since Q is smaller than G (by the assumption of Case 2), Q is 3-colorable. Since a 3-coloring of Q can not be transformed into a 3-coloring of G, we conclude that either z_1 exists and x_1, y_1 are in distinct sets V_1, V_2, or else z_1 does not exist and x_1, y_1 are in the same of the sets V_1, V_2. Therefore, Case 2.2 splits up into Cases 2.2.1, 2.2.2, 2.2.3 below.

Case 2.2.1. z_1 exists. If $x_1 \in V_2$, then any cycle of $G - \{v_0, y_1\}$ through x_1 and e_2 is odd. (Recall that all cycles through e_1 and e_2 are even since otherwise v_0 is of type 1.) But then v_0 cannot be joined to any vertex x' of V_1 (because any cycle of $G-\{v_0,x_1\}$ through x' and e_2 could then be extended to a TOKS in which v_0 is of type 1). But now the above 3-coloring of Q can be extended to G by giving all vertices of V_2 the same color as z_0 and all vertices of $V_1 \setminus \{x_2,y_2\}$ the same color as v_0 and finally coloring z_1, a contradiction. We assume now that $x_1 \in V_1$. Then we may assume that v_0 is joined to no vertex of V_2.

Consider again $H+v_0$. We add the edges v_0x_2, v_0y_2 if they are not already present. Any 3-coloring of the resulting graph can be extended to a 3-coloring of G by first giving all vertices of V_2 the same color as that of v_0 and then giving $V_1 \setminus \{x_2,y_2\}$ another color. So we may assume that $H+v_0+v_0x_2+v_0y_2$ satisfies the conclusion of Theorem 5.1. The configuration implied by Theorem 5.1 may contain the edges v_0x_2, v_0y_2. We replace these edges by two odd paths P_1, P_2 in $B'-x_1$ from y_1 to $\{x_2,y_2\}$, such that $P_1 \cap P_2 = \{y_1\}$ together with the even path $y_1z_1v_0$. The only problem occurs when v_0 is the distinguished vertex of a strong TOKS using precisely one of v_0x_2, v_0y_2, say v_0x_2. (If the strong TOKS uses both of these edges then G has a TOKS with y_1 but not v_0 as a branch vertex in which case v_0 is of type 2.) Consider the case where y_2 is in the strong TOKS even if the edge v_0y_2 is not. In that case we can use the above paths P_1, P_2 and $y_1z_1v_0$ to find a TOKS having y_1 but not v_0 as a branch vertex unless y_2 and x_2 belong to the same cycle C in the strong TOKS in $H+v_0+v_0x_2$ and can interchange roles in the TOKS. (If y_2 is not on the cycle C containing x_2 but a neighbor to C then the strong TOKS in $H+v_0+v_0x_2$ may be replaced by a smaller one in $H+v_0+v_0y_2$.) But now the even path from x_2 to y_2 in the strong TOKS may be replaced by a path in $B'-x_1z_1$ containing x_1 and we obtain a strong TOKS in G. We may therefore assume that y_2 is not contained in the strong TOKS in $H+v_0+v_0x_2$.

By (5.1.1), $H+v_0$ is 3-colorable. If also $H+v_0+v_0y_2$ is 3-colorable, then $H+v_0$ has a 3-coloring such that v_0, x_2, y_2 have colors 1, 1, 2 (because $H+v_0+v_0y_2+v_0x_2$ is not 3-colorable). On the other hand, if $H+v_0+v_0y_2$ is not 3-colorable, then it satisfies the conclusion of Theorem 5.1 and we repeat the argument above with $H+v_0+v_0y_2$ instead of $H+v_0+v_0y_2+v_0x_2$. In that case $H+v_0+v_0y_2$ has a strong TOKS containing y_2 but not x_2, and in any 3-coloring of $H+v_0$, the vertices v_0 and y_2 have the same color. So we may assume that $H+v_0$ has a 3-coloring such that v_0 and x_2 have color 1 and that y_2 has color 2 unless x_2 and y_2 can interchange roles. We now extend the 3-coloring of H_0+v_0 by coloring as many vertices of V_2 as possible by the color 1. What remains of B' cannot be 2-colorable so it must have an odd cycle $C'$and that cycle C' must contain $x_1z_1y_1$ and a vertex w in V_2 joined to a vertex of color 1 since otherwise w would have color 1. Recall that w is not joined to v_0. So w is joined to x_2 (or y_2 if y_2 can play the role of x_2). But now C' can be added to the strong TOKS in $H+v_0+v_0x_2$ to obtain a strong TOKS in G. This contradiction shows that z_1 does not exist. So x_1, y_2 are either both in V_1 or both in V_2.

Case 2.2.2. x_1, $y_1 \in V_1$. We apply the reasoning of the previous case with only minor modifications. We may assume that v_0 is not joined to any vertex of V_2. We may also assume that $H+v_0$ has a 3-coloring. If the color of v_0 is distinct from the colors of x_2, y_2 then we get a 3-coloring of G by giving V_2 the same color as v_0. So assume that v_0 and x_2 have the same color 1 and that y_2 has color 1 or 2. We give all vertices of $V_1 \backslash \{x_1\}$ the color 2 and x_1 the color 3. Since this cannot be extended to a 3-coloring of G, there must be a vertex y_1' in V_2 of degree ≥ 3 joined to x_1 and one of x_2, y_2. Now we repeat the arguments of Case 2.2.1 (after the assumption that $x_1 \in V_1$) where now y_1' replaces y_1 in some of the arguments. For example, $H+v_0+v_0x_2+v_0y_2$ has a strong TOKS with v_0 as distinguished vertex using precisely one of the edges v_0x_2, v_0y_2, say v_0x_2. Moreover, y_2 is not in that strong TOKS. In the 3-coloring of $H+v_0$ we may assume that v_0 and x_2 have the same color 1. (Otherwise we consider $H+v_0+v_0y_2$ and conclude as in Case 2.2.1 that x_2 and y_2 can interchange roles.) We may also assume that y_2 has color 2 unless x_2 and y_2 can interchange roles. Now we color as many vertices of V_2 as possible by color 1. What remains of B' (including y_2 if y_2 has color 2) is not bipartite and we obtain an odd cycle through e_1 which can be added to the strong TOKS as in the previous case. This completes the proof when x_1, $y_1 \in V_1$.

Case 2.2.3. x_1 and y_1 are in V_2. Since all cycles in $G-v_0$ through e_1, e_2 are even and hence all cycles in $G-\{v_0,x_1\}$ through e_2 and y_1 are odd it follows easily that v_0 is not joined to any vertex of V_1. Also the paths in H from x_2 to y_2 containing e_2 are odd. Since $G-v_0$ has no TOKS, we may assume that not all edges from $\{x_1,y_1\}$ to $\{x_2,y_2\}$ are present. So assume that $y_1y_2 \notin E(G)$. We now let H'' be obtained from $H+v_0$ by identifying v_0 and x_2. If $\chi(H'') \geq 4$, then H'' satisfies the conclusion of Theorem 5.1. Then also G does. (If H'' has a strong TOKS and x_2 has degree 1 in the corresponding subgraph in G, then we add a cycle in $B'-y_2$ containing x_2 and e_1.) So we may assume that H'' has a 3-coloring, i.e., $H+v_0$ has a 3-coloring such that both v_0 and x_2 have color 1. Assume that y_2 has color 1 or 2. Now we obtain a 3-coloring of G by coloring y_1 by 2, all vertices of $V_1 \setminus \{y_2\}$ by 1, and all vertices of $V_2 \setminus \{y_1\}$ by 3, a contradiction which completes the proof of (5.1.7).

We may assume that $N(v_0)$ does not contain a 3-cycle since otherwise v_0 is of type 1. It now follows from (5.1.7) that $N(v_0)$ contains a vertex w_0 such that all edges of $N(v_0)$ (if any) are incident with w_0. Let G_1 be obtained from G by deleting v_0 and identifying $N(v_0) \setminus \{w_0\}$ into a single vertex v_1. Clearly, $\chi(G_1) \geq 4$. By the minimality property of G, G_1 satisfies Theorem 5.1 with v_1 instead of v_0. If v_1 is of type 0, 2 or 3, then clearly, v_0 is of type 0, 2 or 3 in G. So, we may assume that v_1 is the distinguished vertex of a strong TOKS in G_1. Consider first the case where v_1 is of type 1, that is, $G_1 - v_1$ has a cycle C_1 such that v_1 is joined to three vertices x_1, x_2, x_3 on C_1, and $C_1[x_1,x_2]$, $C_1[x_2,x_3]$, $C_1[x_2,x_3]$ are odd. Let y_1, y_2, y_3 be vertices in $N(v_0) \setminus \{w_0\}$ such that $x_i y_i \in E(G)$ for $i = 1$, 2, 3. We consider two cases.

Case 1. $G - (V(C_1) \cup \{v_0\})$ has no path joining any two of y_1, y_2, y_3. In particular y_1, y_2, y_3 are distinct. We shall now apply Proposition 4.2. If $G - v_0 - x_i$ has an odd path from y_i to C_1 for $i = 1$, 2, 3, then Proposition 4.2 implies that v_0 is of type 0, 2 or 3. So we may assume by Proposition 4.2 that all paths in $G - v_0 - x_1$ from y_1 to C_1 are even. More generally, we may assume that there is no odd subdivision of a 1-extension or 2-extension or 3-extension from y_1 to C_1, v_0. Consider the component Q_1 of $G - (V(C_1) \cup \{v_0\})$ containing y_1.

Case 1.1. Q_1 is bipartite with bipartition V_1, V_2 where $y_1 \in V_1$. Then no vertex in V_1 has a neighbor in $C_1 - x_1$. Since $G - [V(Q_1) \setminus \{y_1\}]$ is not a counterexample to Theorem 5.1, it has a 3-coloring. We extend that 3-coloring by giving all vertices of V_1 the same color as y_1. Since that cannot be extended to a 3-coloring of G, there is a vertex u_1 in V_2 which has three neighbors of different colors. If two of these vertices are in C_1, then a path in Q_1 from y_1 to u_1 together with two edges from u_1 to C_1 is an odd subdivision of a 2-extension from y_1 to C_1. So assume that u_1 has only one neighbor u_1' in C_1. Then also u_1 is joined to v_0. If $u_1' = x_1$, then u_1 can play the role of y_1 and hence no vertex in V_2 has a neighbor in $C_1 - x_1$ contradicting the assumption that $G - v_0$ is 2-connected. So $u_1' \neq x_1$. Since $[G - V(Q_1)] \cup \{x_1 u_1'\}$ is not a counterexample to Theorem 5.1 it has a 3-coloring. We extend that 3-coloring by giving all vertices in V_1 the same color, namely the same color as u_1' (unless that color is also the color of v_0). Since that 3-coloring cannot be extended to G we may assume that there is a vertex u_2 in V_2 joined to vertices of all three colors such that precisely one of these is a vertex u_2' in C_1 and another is v_0. Clearly $u_2' \neq u_1'$. Now we consider a minimal connected subgraph in Q_1 containing y_1, u_1, u_2 and we obtain an odd subdivision of a 2-extension or a 3-extension from y_1 to C_1 a contradiction. This completes the proof when $G-(V(C_1)\cup\{v_0\})$ has no path between any two of y_1, y_2, y_3 and Q_1 is bipartite.

Case 1.2. $G-(V(C_1)\cup\{v_0\})$ has no path joining any two of y_1, y_2, y_3, all paths in $G-v_0-x_1$ from y_1 to C_1 are even, and the above component Q_1 is not bipartite. Then we let Q_2 be the unique maximal connected subgraph of Q_1 such that Q_2 contains y_1, and Q_2 is the union of bipartite blocks in Q_1. Since $G-\{v_0,x_1\}$ has a path from y_1 to C_1 and all such paths are even, Q_2 is nonempty. Let V_1, V_2 be the bipartition of Q_2, where $y_1\in V_1$. Then no vertex in V_1 has a neighbor in C_1-x_1. Let u_1 be a vertex in Q_1 having a neighbor u_1' in C_1-x_1. Then u_1 is in V_2, and we say that u_1 is *nice*. We may assume that every nice vertex has precisely one neighbor in C_1 since otherwise there is an odd subdivision of a 2-extension from y_1. Any vertex in V_1 which is joined to both x_1 and v_0 is called *neat*. (Note that no vertex in V_2 can be joined to both x_1 and v_0 since otherwise that vertex in V_2 could play the role of y_1.) Finally, a vertex in $V_1\cup V_2$ is *awkward* if it is contained in a nonbipartite block of Q_1. If Q_2 has two paths P_1, P_2 from a nice vertex (say u_1) to a neat vertex (say y_1) and an awkward vertex (say z_1), respectively, such that $P_1\cap P_2=\{u_1\}$ (possibly $P_2=u_1$), then it is easy to find an odd subdivision of a 1-extension or 2-extension from y_1 because Q_1 has a path P_3 of any prescribed parity from z_1 to a neighbor of C_1 such that $P_3\cap Q_2=\{z_1\}$. So we may assume that no such paths P_1, P_2 exist. But then Q_2 has a cutvertex z_2 such that some component Q_3 of Q_2-z_2 contains a nice vertex (say u_1) but no neat vertex and no awkward vertex. Choose z_2 such that Q_3 is smallest possible. Then a vertex u_2 in $V_2\cap[V(Q_3)\cup\{z_2\}]$ may be joined to u_1' but to no other vertex on C_1. For otherwise, the minimality of Q_3 implies that Q_2 has two paths P_1, P_2 from u_1 to u_2 and z_2, respectively, such that $P_1\cap P_2=\{u_1\}$ (or $P_1=P_2$ if $u_2=z_2$), and hence there is an odd subdivision of a 2-extension from y_1. We may also assume that no nice vertex w in Q_3 is joined to v_0. For otherwise we obtain an odd subdivision of a 1-extension from w by taking a path in Q_2 from w to an awkward vertex and then extending that to an odd path from w to x_1 using a nonbipartite block of Q_1. (Note that the path has to end at x_1 on C_1 since all paths in $G-v_0-x_1$ from y_1 to C_1 have the same parity.) Now we delete Q_3 from G. If $z_2\in V_1$ and some vertex of $V(Q_3)\cap V_1$ is joined to x_1 we also add the edge z_2x_1. If $z_2\in V_2$ we add instead the edge z_2u_1'. By the minimality of G, the resulting graph has a 3-coloring. We now obtain a contradiction by extending the resulting 3-coloring to G as follows. Let $i\in\{1,2\}$ be such that $z_2\in V_i$. If possible, we give all vertices of $V_i\cap V(Q_3)$ the same color as z_2, and then we can color the vertices in $V_{3-i}\cap V(Q_3)$ successively. If this is not possible, then z_2 and v_0 have the same color. But then we can give all vertices of $V_{3-i}\cap V(Q_3)$ the same color namely a color distinct from that of z_2 and u_1' if $i=1$ or that of z_2 and x_1 if $i=2$, and then color the vertices in $V_i\cap V(Q_3)$ successively. This completes the proof when $G-(V(C_1)\cup\{v_0\})$ has no path between any two of y_1, y_2, y_3.

Case 2. $G-(V(C_1)\cup\{v_0\})$ has a path P_0 (possibly of length 0) between y_2 and y_3 such that $y_1\notin V(P_0)$. If P_0 is even, then G has a TOKS containing $C_1\cup P_0$ and with branch vertices x_1, x_2, x_3, y_3. So we may assume that P_0 is odd. If $G-(V(C_1)\cup\{v_0\})$ has a path P_1 from y_1 to P_0, then the notation can be chosen such that $P_0\cup P_1$ contains an even path P_2 from y_1 to y_3. If P_2 does not contain y_2, then we obtain a contradiction as in the above case where P_0 was even. On the other hand, if P_2 contains y_2, then G has a TOKS with branch vertices v_0, y_1, y_2, y_3 showing that v_0 is of type 3. So we may assume that $G-(V(C_1)\cup\{v_0\})$ has no path from y_1 to P_0. We now define Q_1 as the component of $G-(V(C_1)\cup\{v_0\})$ containing y_1 and we repeat the reasoning from the previous paragraphs (Case 1) and we complete the proof by applying Proposition 4.1 rather than Proposition 4.2. This completes the proof if v_1 is of type 1 in G_1.

Let H_1 denote the strong TOKS in G_1 having v_1 as the distinguished vertex. Choose H_1 such that H_1-v_1 has as few cycles as possible. We may assume that H_1-v_1 has at least two cycles. Let C_1 be a cycle which is an endblock of H_1-v_1. Let H be the subgraph in G corresponding to H_1 (such that no vertex in H has degree 0). If some vertex in H but not H_1-v_1 has degree at least 2 in H, then it is easy to find a TOKS showing that v_0 is of type 0 or 2. So we may assume that each vertex in H corresponding to v_1 in H_1 has degree 1 in H. Let y_1, y_2 denote the two vertices of degree 1 in H joined to say x_1 and x_2 (respectively) in C_1. Let z_3 denote the unique vertex in H_1-v_1 which is not in C_1 but joined to a vertex x_3 in C_1. Let C_2 denote the cycle in H_1-v_1 containing z_3, and let z_1, z_2 be the two other vertices on C_2 having neighbors (in H) u_1, u_2 (respectively) that are not in C_2.

Let P_1 denote a path in $G-v_0-x_1$ from y_1 to $H-y_1$. If the other end of P_1 is in the component of $H-z_3$ containing C_2-z_3, then G has a TOKS with branch vertices y_1, x_1, x_2, x_3, a contradiction. For the same reason, P_1 must be odd if its other end is z_3. If the other end of P_1 is z_3, then we also consider a path P_2 in $G-v_0-x_2$ from y_2 to $H\cup P_1-y_2$. The other end of P_2 must be on $P_1\cup C_1$. If the other end of P_2 is on P_1, then it is easy to show that v_0 is of type 2 or 3, a contradiction. So the other end of P_2 must be on C_1. Then we repeat the argument of the previous case (where v_1 is of type 1) to conclude that $G-v_0$ has an odd subdivision of a j-extension ($j=1$, 2 or 3) from y_2 to C_1. By Proposition 4.1, G has the desired TOKS, a contradiction. So, P_1 must have an end on C_1+y_2. If an end of P_1 is y_2, then P_1 must be odd since otherwise G has a TOKS with branch vertices y_1, x_1, x_2, x_3. If P_1 cannot be chosen such that y_2 is an end, then $G-v_0$ has an odd subdivision of a j-extension ($j=1$, 2 or 3) from y_i to C_1 for $i=1$, 2 such that these extensions do not intersect the component of $H-x_3$ containing z_3.

Now we focus on z_3. Consider the case where $G-v_0-x_3-z_3$ has no path joining the two components of $H-x_3-z_3$. In that case every path in $G-v_0-x_3$ joining the two components of $H-x_3$ (and such paths do exist) must have z_3 as an end. But then we repeat the argument of the case where v_1 is of type 1 to conclude that $G-v_0$ has an odd subdivision of a j-extension from z_3 to C_1 containing no vertex of $H-(C_1+y_1+y_2+z_3)$. (We treat z_3 as a vertex which is both awkward and neat.) Then the proof is completed using Propositions 4.1 or 4.2. We may therefore assume that $G-v_0-x_3-z_3$ has a path Q joining a vertex w_1 of $C_1+y_1+y_2$ with a vertex w_2 in the other component of $H-x_3-z_3$. We may assume that w_1 is in C_1. We may also assume that w_2 is in C_2 for otherwise it is easy to find a TOKS containing C_2 having z_1, z_2, z_3 and one of w_1, w_2, x_3 as branch vertices.

If C_2 has a path Q' from z_3 to w_2 such that Q' contains at most one of z_1, z_2 and $Q'\cup Q$ is odd, then we consider an even path Q'' in H from z_3 to a neighbor of v_0 such that $Q''\cap Q'=\{z_3\}$ and such that Q'' contains a subpath in C_2 from z_3 to one of z_1, z_2. We complete the proof by Proposition 4.1 or 4.2 since $Q\cup Q'\cup Q''\cup\{z_3x_3\}$ is an odd subdivision of a 1-extension from z_3 to C_1. So we may assume that $w_2\in C_2[z_3,z_1]$ and that $C_2[z_3,w_2]$ and Q have the same parity. The minimality of H_1 implies that $C_1\cup C_2\cup Q\cup\{x_3z_3\}$ does not contain an odd cycle which is divided into three odd paths by three of z_1, z_2, x_1, x_2. This implies that w_1 is in $C_1[x_3,x_1]\cup C_1[x_3,x_2]$, $w_1\neq x_1,x_2$ and that $C_1[x_3,w_1]$ is odd. But then $C_2[z_3,w_2]\cup Q$ is even and can be extended to an odd subdivision of a weak 2-extension from z_3 to C_1. Now the proof is completed by Proposition 4.2. ∎

Acknowledgement. Thanks are due to Tommy Jensen for his careful reading and numerous comments.

References

[1] J. A. Bondy and L. Lovász: Cycles through specified vertices of a graph, *Combinatorica,* **1** (1981), 117–140.

[2] P. A. Catlin: Hajós graph-colouring conjecture: variations and counterexamples, *J. Combinatorial Theory Ser. B*, **26** (1979), 268–274.

[3] G. A. Dirac: A property of 4-chromatic graphs and some remarks on critical graphs, *J. London Math. Soc.,* **27** (1952), 85–92.

[4] H. Hadwiger: Über eine Klassifikation der Streckenkomplexe, *Vierteljahrsschrift der naturforschenden Gesellschaft in Zürich,* **88** (1943), 133–142.

[5] T. R. Jensen and F. B. Shepherd: Note on a conjecture of Toft, *Combinatorica,* **15** (1995), 373–377.

[6] T. R. Jensen and B. Toft: *Graph Coloring Problems,* John Wiley, New York, 1995.

[7] L. Lovász: Problem 5, *Period. Math. Hungar.,* **4** (1974), 82.

[8] W. D. McCuaig and M. Rosenfeld: Parity of cycles containing specified edges, in: Cycles in Graphs (B. R. Alspach and C. D. Godsil, eds.), *Annals of Discrete Math.,* **27** (1985), 419–431.

[9] C. Thomassen: Planarity and duality of finite and infinite graphs, *J. Combinatorial Theory Ser. B,* **29** (1980), 244–271.

[10] C. Thomassen: Graph decomposition with applications to subdivisions and path systems modulo k, *J. Graph Theory,* **7** (1983), 261–271.

[11] C. Thomassen: Plane representations of graphs, in: *Progress in Graph Theory* (J. A. Bondy and U. S. R. Murty, eds.), Academic Press, Toronto 1984, 43–69.

[12] B. Toft: Problem 11, in: *Recent Advances in Graph Theory,* Academia Praha 1975, 543–544.

[13] K. Wagner: Über eine Eigenschaft der ebenen Komplexe, *Math. Ann.,* **114** (1937), 570–590.

[14] W. Zang: Proof of Toft's Conjecture: Every graph containing no fully odd K_4 is 3-colorable, *J. Combin. Optimization,* **2** (1998) 117–188.

Carsten Thomassen

Department of Mathematics
Technical University of Denmark
DK-2800 Lyngby, Denmark
C.Thomassen@mat.dtu.dk